Indeed, if $v(t) = v_0 e^{-t}$, then $v(t)$ is a decreasing function , $\int_0^\infty v(t)dt < \infty$, and $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = 0$. So $0 \in C$.

If $a \geq 1$ and $v(t) = \frac{v_0}{1 + t^a \ln^2(1+t)}$ then $v(t)$ is a decreasing function, $\int_0^\infty v(t)dt < \infty$ and

$\lim_{t\to\infty} \frac{v(2t)}{v(t)} = \lim_{t\to\infty} \frac{1 + t^a \ln^2(1+t)}{1 + 2^a t^a \ln^2(1+2t)} = \frac{1}{2^a}$. So $\left(0, \frac{1}{2}\right] \subset C$. It remains to prove that if $c > \frac{1}{2}$ then c$\notin C$.

Suppose if possible that $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = c$, where $c > \frac{1}{2}$. Let $\epsilon := \frac{c - 1/2}{2} > 0$. Then there is a number $T = T(\epsilon) > 0$ such that $-\epsilon < \frac{v(2t)}{v(t)n} - c < \epsilon$, whenever $t > T$. We conclude that

$$\int_{2T}^\infty v(rt)dt = 2\int_T^\infty v(2t)dt > 2(c-\epsilon)\int_T^\infty v(t)dt \geq 2(c-\epsilon)\int_{2T}^\infty v(t)dt = (1+2\epsilon)\int_{2T}^\infty v(t)dt > \int_{2T}^\infty v(t)dt,$$

which is a contradiction, and the proof is complete.

Also solved by the proposer.

5508: *Proposed by Pedro Pantoja, Natal RN, Brazil*

Let a, b, c be positive real numbers such that $a + b + c = 1$. Find the minimum value of

$$f(a,b,c) = \frac{a}{3ab + 2b} + \frac{b}{3bc + 2c} + \frac{c}{3ca + 2a}.$$

Solution 1 by Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we note that since $a, b, c > 0$ and $a + b + c = 1$, the Arithmetic - Geometric Mean Inequality implies that

$$\begin{aligned} a^2 + b^2 + c^2 &= (a+b+c)\left(a^2+b^2+c^2\right) \\ &= a^3 + b^3 + c^3 + ab^2 + bc^2 + ca^2 + a^2b + b^2c + c^2a \\ &= \left(a^3 + ab^2\right) + \left(b^3 + bc^2\right) + \left(c^3 + ca^2\right) + a^2b + b^2c + c^2a \\ &\geq 2\sqrt{a^4b^2} + 2\sqrt{b^4c^2} + 2\sqrt{c^4a^2} + a^2b + b^2c + c^2a \\ &= 3\left(a^2b + b^2c + c^2a\right). \qquad (1) \end{aligned}$$

As a result of (1), we have

$$\begin{aligned} 1 &= (a+b+c)^2 \\ &= a^2 + b^2 + c^2 + 2(ab + bc + ca) \\ &\geq 3\left(a^2b + b^2c + c^2a\right) + 2(ab + bc + ca) \\ &= \left(3a^2b + 2ab\right) + \left(3b^2c + 2bc\right) + \left(3c^2a + 2ca\right). \qquad (2) \end{aligned}$$

Then, using property (2), the convexity of $g(x) = \dfrac{1}{x}$ on $(0, \infty)$, and Jensen's Theorem, we obtain

$$\begin{aligned}
f(a,b,c) &= \frac{a}{3ab+2b} + \frac{b}{3bc+2c} + \frac{c}{3ca+2a} \\
&= ag(3ab+2b) + bg(3bc+2c) + cg(3ca+2a) \\
&\geq g\left[a(3ab+2b) + b(3bc+2c) + c(3ca+2a)\right] \\
&= g\left[\left(3a^2b+2ab\right) + \left(3b^2c+2bc\right) + \left(3c^2a+2ca\right)\right] \\
&= \frac{1}{(3a^2b+2ab)+(3b^2c+2bc)+(3c^2a+2ca)} \\
&\geq 1 \\
&= f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).
\end{aligned}$$

It follows that under the conditions $a, b, c > 0$ and $a + b + c = 1$, the minimum value of $f(a,b,c)$ is $f\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) = 1$.

Solution 2 by David E. Manes, Oneonta, NY

We will show that the minimum value of f is 1.
By the Arithmetic Mean-Geometric Mean inequality, we get

$$f(a,b,c) \geq 3\sqrt[3]{\frac{a}{b(3a+2)} \cdot \frac{b}{c(3b+2)} \cdot \frac{c}{a(3c+2)}} = \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}}.$$

We again use the AM-GM inequality to obtain

$$\begin{aligned}
\sqrt[3]{(3a+2)(3b+2)(3c+2)} &\leq \frac{(3a+2)+(3b+2)+(3c+2)}{3} = \frac{3(a+b+c)+6}{3} \\
&= 3.
\end{aligned}$$

Hence,

$$\frac{1}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq \frac{1}{3}$$

so that

$$f(a,b,c) \geq \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq 3 \cdot (1/3) = 1$$

with equality if and only if $a = b = c = \dfrac{1}{3}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

From Bergström's and the Arithmetic mean -Geometric mean inequalities,

$$f(a,b,c) = \frac{\left(\sqrt{\frac{a}{b}}\right)^2}{3a+2} + \frac{\left(\sqrt{\frac{b}{c}}\right)^2}{3b+2} + \frac{\left(\sqrt{\frac{c}{a}}\right)^2}{3c+2} \geq \frac{\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}\right)^2}{3a+2+3b+2+3c+2} = \left(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}}{3}\right)^2$$

$$\geq \sqrt[3]{\sqrt{\frac{a}{b}}\sqrt{\frac{b}{c}}\sqrt{\frac{c}{a}}} = 1.$$

Equality is attained iff it occurs in those two inequalities, that is, iff $\dfrac{\sqrt{\frac{a}{b}}}{3a+2} = \dfrac{\sqrt{\frac{b}{c}}}{3b+2} = \dfrac{\sqrt{\frac{c}{ba}}}{3c+2}$ and $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{a}$. These last identities are true if and only if $a = b = c$, that is, if and only if $a = b = c = \dfrac{1}{3}$. In this case equality is also obtained in Bergström's inequality. So, the minimum value of $f(a,b,c)$ is 1, and this occurs if and only if $a = b = c = \dfrac{1}{3}$.

Solution 4 by Arkady Alt, San Jose,CA

Since $\left(\dfrac{a}{3a+2} - \dfrac{b}{3b+2}\right)\left(\left(-\dfrac{1}{a}\right) - \left(-\dfrac{1}{b}\right)\right) = \dfrac{2(a-b)^2}{ab(3b+2)(3a+2)} \geq 0$ then triples $\left(\dfrac{a}{3a+2}, \dfrac{b}{3b+2}, \dfrac{c}{3c+2}\right)$, $\left(-\dfrac{1}{a}, -\dfrac{1}{b}, -\dfrac{1}{c}\right)$ are agreed in order and, therefore, by the Rearrangement Inequality $\sum\limits_{cyc} \dfrac{a}{3a+2} \cdot \left(-\dfrac{1}{a}\right) \geq \sum\limits_{cyc} \dfrac{a}{3a+2} \cdot \left(-\dfrac{1}{b}\right) \iff$

$$\sum_{cyc} \frac{a}{(3a+2)\,b} \geq \sum_{cyc} \frac{a}{3a+2} \cdot \frac{1}{a} = \sum_{cyc} \frac{1}{3a+2}.$$

Also, by Cauchy Inequality $\sum\limits_{cyc}(3a+2) \cdot \sum\limits_{cyc} \dfrac{1}{3a+2} \geq 9 \iff 9 \cdot \sum\limits_{cyc} \dfrac{1}{3a+2} \geq 9 \iff \sum\limits_{cyc} \dfrac{1}{3a+2} \geq 1$.Thus, $f(a,b,c) \geq 1$ and since $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$ we may conclude that $\min f(a,b,c) = 1$.

Solution 5 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Since $c = 1 - a - b$, then we have:

$$f(a,b,c) = \frac{a}{3ab+2b} + \frac{b}{3b(1-a-b)+2(1-a-b)} + \frac{1-a-b}{3(1-a-b)a+2a}.$$

That means that we may assume the function:

$$g(a,b) = \frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b+-1}{a(3a+3b-5)}.$$

To find the stationary points of $g(a,b)$, work out $\dfrac{\partial g}{\partial a}$ and $\dfrac{\partial g}{\partial b}$ and set both to zero . This gives two equations for two unknowns a and b. We may solve these equations for a and b (often there is more than one solution). Let (x,y) be a stationary point. If $g_{aa} > 0$ and $g_{bb} > 0$ at (x,y) then (x,y) is a minimum point . So,

$$\frac{\partial g}{\partial a} = -\frac{(a+b-1)(6a+3b-5)}{a^2(3a+3b-5)^2} - \frac{3a}{b(3a+2)^2} + \frac{b}{(3b+2)(a+b-1)^2} + \frac{1}{3ab+2b} + \frac{1}{a(3a+3b-5)}$$

$$\frac{\partial g}{\partial b} \;=\; -\frac{a}{(b^2(3a+2)} + \frac{b(3a+6b-1)}{(3b+2)^2(a+b-1)^2} - \frac{1}{(3b+2)(a+b-1)} + \frac{1}{a(3a+3b-5)} - \frac{3(a+b-1)}{a(3a+3b-5)^2},$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min \left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min \left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

Solution 6 by Albert Stadler, Herrliberg, Switzerland

We will prove that the minimum value equals 1 and the minimum is assumed if and only if $a = b = c = 1/3$. To that end we must prove that

$$f(a,b,c) = \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} \geq 1.$$

We clear denominators and get the equivalent inequality

$$10\sum_{cycl} a^4b^2 + 24\sum_{cycl} a^3b^3 + 18\sum_{cycl} a^4c^2 + 4\sum_{cycl} a^4c^2 \geq 2\sum_{cycl} a^4bc + 15\sum_{cycl} a^4b^2c + 11\sum_{cycl} a^4bc^2 + 28\sum_{cycl} a^2b^2c^2. \quad (1)$$

By the (weighted)AM-GM inequality,

$$\sum_{cycl} a^4b^2 + \sum_{cycl} a^4c^2 \geq 2\sum_{cycl} a^4bc,$$

$$15\sum_{cycl} a^3b^3 = 15\sum_{cycl}\left(\frac{2}{3}a^3b^3 + \frac{1}{3}c^3a^3\right) \geq 15\sum_{cycl} a^3b^2c,$$

$$11\sum_{cycl} a^4c^2 = 11\sum_{cycl}\left(\frac{2}{3}a^4c^2 + \frac{1}{6}b^4a^2 + \frac{1}{6}c^4b^2\right) \geq 11\sum_{cycl} a^3bc^2,$$

$$9\sum_{cycl} a^4b^2 \geq 27a^2b^2c^2,$$

$$9\sum_{cycl} a^3b^3 \geq 27a^2b^2c^2,$$

$$6\sum_{cycl} a^4c^2 \geq 18a^2b^2c^2,$$

$$4\sum_{cycl} a^5c \geq 12a^2b^2c^2,$$

and (1) follows if we add the last seven inequalities. In all seven inequalities equality holds if and only if $a = b = c$.

Comment by Stanley Rabinowitz of Chelmsford, MA. Problems such as this are easily solvable by computer algebra systems these days. For example; the Mathematica command

$$\text{Minimize}\,[\{\text{a}/(3\text{a}*\text{b}+2\text{b})+\text{b}/(3\text{b}*\text{c}+2\text{c})+\text{c}/(3\text{c}*\text{a}+2\text{a}),\ \text{a}>0\&\&\ \text{b}>0\&\&\text{c}>0\&\&\{\text{a}+\text{b}+\text{c}=1\},\{\text{a},\text{b}$$

responds by saying that the minimum value is 1 and occurs when $a = b = c = \dfrac{1}{3}$.

Also solved by Konul Aliyeva (student), ADA University, Baku, Azerbaijan; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Tran Hong (student), Cao Lanh School, Dong Thap, Vietnam; Sanong Huayrerai, Rattanakosinsomphothow School, Nakon, Pathom, Thailand; Seyran Ibrahimov, Baku State University, Maasilli, Azerbaijan; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Stanley Rabinowitz of Chelmsford, MA; Neculai Stanciu "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Daniel Văcaru, Pitesti, Romania; Nicusor Zlota "Traian Vuia Technical College, Focsani, Romania, and the proposer.

5509: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers that add up to one and such that $0 < \dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x} < \dfrac{\pi}{2}$. Prove that

$$\sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)} < \frac{3}{5}\sqrt{5}.$$

Solution 1 by Michel Bataille, Rouen, France

The Cauchy-Schwarz inequality provides

$$\sqrt{x}\sqrt{\cos\left(\frac{y}{z}\right)}+\sqrt{y}\sqrt{\cos\left(\frac{z}{x}\right)}+\sqrt{z}\sqrt{\cos\left(\frac{x}{y}\right)} \le (x+y+z)^{1/2}\left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Since $x + y + z = 1$, it follows that the left-hand side L of the proposed inequality satisfies

$$L \le \left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Thus, it suffices to show that

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right) < \frac{9}{5}. \tag{1}$$

Now, Jensen's inequality applied to the cosine function, which is concave on $(0, \frac{\pi}{2})$, yields

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right) \le 3\cos\left(\frac{y/z+z/x+x/y}{3}\right). \tag{2}$$